Inside a « magnetically » empty region of space:

$$\vec{B} = \mu_0 \vec{H} = -\mu_0 \overrightarrow{\mathrm{grad}} V^*$$

$$\Delta V^* = 0 \ \rightarrow \Delta\left(B_x, B_y, B_z, \rho B_\varphi, \ldots\right) = 0$$

Laplace's equation is separable in 13 coordinate systems.
Every quantity obeying Laplace's equation can be expanded in a
unique Laplace series,
Walter Rudin, PhD Thesis, Duke University, 1949,
« Uniqueness Theory for Laplace Series »,
Trans. Amer. Math. Soc, Vol. 68, N°2, March 1950, pp. 287-303.

Spherical coordinates,
ball of center O and radius r_{max} « magnetically » empty

$$\Delta B_z = 0$$

$$\frac{B_z(r,\vartheta,\varphi)}{B_0} = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n \left[Z_n P_n(\cos\vartheta) + \sum_{m=1}^{n}\left(X_n^m \cos m\varphi + Y_n^m \sin m\varphi\right) W_n^m P_n^m(\cos\vartheta) \right]$$

$$\left|W_n^m P_n^m(\cos\vartheta)\right| \le 1 \rightarrow W_n^m = \frac{(n-m-1)!!}{(n+m-1)!!}$$

$$Z_n, X_n^m, Y_n^m \propto \left(\frac{r_0}{r_{max}}\right)^n$$

Unique set of coefficients

Spherical Harmonics Expansion (SHE)

1. To what maximum degree *n* and maximum order *m* can the SHE be limited while ensuring that the quantity is described within a given accuracy throughout the domain?

2. How can it be demonstrated from the measurements themselves that these limits are acceptable?

3. The number of measurement points must be at least equal to the number of SHE coefficients to be determined. Then, how are they to be distributed for making this determination as insensitive as possible to the measurement accuracy both in value and in position?

Why considering a single component of the field?

If B_z is the « principal » component, then:

Ideal $\rightarrow \vec{B} = B_0 \overrightarrow{u_z}$

$$\vec{B} = B_0 \overrightarrow{u_z} + \vec{b} \quad \vec{b} = b_x \overrightarrow{u_x} + b_y \overrightarrow{u_y} + b_z \overrightarrow{u_z} \quad \left|\vec{b}\right| \ll B_0$$

$$\nu_{RMN} \propto \left|\vec{B}\right| = \sqrt{b_x^2 + b_y^2 + \left(B_0 + b_z\right)^2} \simeq B_0 \left[1 + \frac{b_z}{B_0} + O\left(\frac{b}{B_0}\right)^2\right]$$

2. The Condon–Shortley phase factor

In order to understand the genesis of the above-mentioned confusion (Section II A 1), one must read §4 "Orbital angular momentum" of their book.[2] They first find the solution numbered (10) for $m = n$, written in their notations (changing only l in n and m_l in m)

$$\Theta(n,n) = (-1)^n \sqrt{\frac{(2n+1)!}{2}} \frac{1}{2^n n!} \sin^n \vartheta, \tag{9}$$

with the comment: "*The phase has been taken as $(-1)^n$ for convenience in later work*". The demonstration then leads to the results numbered (17) and (18)

$$\begin{aligned} \Theta(n,m) &= (-1)^m \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}} \sin^m \vartheta \frac{d^m}{(d\cos\vartheta)^m} P_n(\cos\vartheta), \\ \Theta(n,-m) &= +\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}} \sin^m \vartheta \frac{d^m}{(d\cos\vartheta)^m} P_n(\cos\vartheta), \\ \Theta(n,-m) &= (-1)^m\, \Theta(n,m), \;\; m > 0, \end{aligned} \tag{10}$$

with the comment: "*The natural choice of phases which we have made here leads to a rather curious occurrence of the factor −1 only for positive odd values of m. If we had approached the problem through the usual form of the theory of spherical harmonics the natural tendency would have been to choose the normalizing factors with omission of the $(-1)^m$ in these formulas*". It is thus clear that they did not intend to include this factor in the Rodrigues' formula which was stillthe case for some of the authors.

For our part, we shall leave this factor to the CSH and keep the associated Legendre functions in their original form which gives $n = m = 1$:

$$\begin{aligned} Y_1^1(\vartheta,\varphi) &= -\sqrt{\frac{3}{8\pi}} \sin\vartheta \exp(i\varphi), \\ P_1^1(\cos\vartheta) &= +\sin\vartheta. \end{aligned} \tag{11}$$

2. Weighting vs. normalization

From a mathematical viewpoint, there is no need for extra coefficients to transform the above defined functions under a rotation of the reference frame. As already mentioned,this is not the case for numerical evaluations for the stability of recurrences. Indeed, if the range of the Legendre polynomials $P_n(\cos\vartheta)$ is strictly $[-1,+1]$, that of the associated Legendre functions $P_n^m(\cos\vartheta)$ increases very fast with their degree and order as a consequence of the factorials in relation (3). The normalization coefficient N_n^m of relation (7) overcomes this drawback for the CSH and could also be used for the RSH but a more elegant solution has been proposed by A. Schmidt.[13] It is known in the geophysics community as the Schmidt quasi-normalization, replacing the original Legendre functions by the following:

$$P_{Sn}^m(\cos\vartheta) = \sqrt{(2-\delta_m^0)\frac{(n-m)!}{(n+m)!}}\, P_n^m(\cos\vartheta), \;\; 0 \le m \le n. \tag{14}$$

One recognizes a part of the CSH normalization coefficient but this definition leaves unchanged the Legendre polynomials ($m = 0$) and, as we shall see later, brings important simplifications in several formulas. As an example, already emphasized by Schmidt, the "addition theorem" of Legendre polynomials takes a much simpler form than the original (4):

$$P_{Sn}^0(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\cos\psi) = \sum_{m=0}^{n} P_{Sn}^m(\cos\vartheta_1) P_{Sn}^m(\cos\vartheta_2)\cos m\psi. \tag{15}$$

$$n_0+1 \quad Z_n$$

$$\sum_{n=1}^{n_0} n = \frac{n_0(n_0+1)}{2} \qquad X_n^m$$

$$\sum_{n=1}^{n_0} n = \frac{n_0(n_0+1)}{2} \qquad Y_n^m$$

$$\rightarrow N = (n_0+1)^2$$

$$m_0(n) = m_0(n_0-n) = n, \quad n = 1, \ldots, \left\lfloor \frac{n_0}{2} \right\rfloor$$

$$\rightarrow N = n_0 + 1 + 2\left\lfloor \frac{n_0}{2} \right\rfloor \left\lfloor \frac{n_0+1}{2} \right\rfloor = \left\lfloor \frac{(n_0+1)^2+1}{2} \right\rfloor$$

$$n_c(n) = \left\lfloor \frac{n^2}{2} \right\rfloor + n + 1$$

$$i_c(n,m) = \left\lfloor \frac{(n+m)^2}{2} \right\rfloor + n - m + 1$$

$$j_c(n,m) = i_c(n,m) + 1$$

$n_0 = 13 \rightarrow$

0: 1						
1: 2	1: 3					
	1: 4					
2: 5	1: 6	2: 9				
	1: 7	2: 10				
3: 8	1: 11	2: 14	3: 19			
	1: 12	2: 15	3: 20			
4: 13	1: 16	2: 21	3: 26	4: 33		
	1: 17	2: 22	3: 27	4: 34		
5: 18	1: 23	2: 28	3: 35	4: 42	5: 51	
	1: 24	2: 29	3: 36	4: 43	5: 52	
6: 25	1: 30	2: 37	3: 44	4: 53	5: 62	6: 73
	1: 31	2: 38	3: 45	4: 54	5: 63	6: 74
7: 32	1: 39	2: 46	3: 55	4: 64	5: 75	6: 86
	1: 40	2: 47	3: 56	4: 65	5: 76	6: 87
8: 41	1: 48	2: 57	3: 66	4: 77	5: 88	
	1: 49	2: 58	3: 67	4: 78	5: 89	
9: 50	1: 59	2: 68	3: 79	4: 90		
	1: 60	2: 69	3: 80	4: 91		
10: 61	1: 70	2: 81	3: 92			
	1: 71	2: 82	3: 93			
11: 72	1: 83	2: 94				
	1: 84	2: 95				
12: 85	1: 96					
	1: 97					
13: 98						

$$Z_{2p},\, X_{2p}^{2q} \neq 0$$

$$\forall p \geq 2,\, q \leq p$$

Aliasing
(folding)

$$\approx$$

Wagon-wheel
effect

COEFFICIENTS DU DHS DE Bx@r0=0.050 (PREMIER EN T, LES AUTRES EN ppm) :

0:	1.000000												
1:	-0.946	1:	0.000										
		1:	0.000										
2:	0.055	1:	0.000	2:	0.015								
		1:	0.000	2:	0.000								
3:	-1.568	1:	0.000	2:	-1.199	3:	0.000						
		1:	0.000	2:	0.000	3:	0.000						
4:	-24038.780	1:	0.000	2:	12019.385	3:	0.000	4:	7110.306				
		1:	0.000	2:	0.000	3:	0.000	4:	0.000				
5:	-2.840	1:	0.000	2:	11.450	3:	0.000	4:	-5.370	5:	0.000		
		1:	0.000	2:	0.000	3:	0.000	4:	0.000	5:	0.000		
6:	900.350	1:	0.000	2:	-562.629	3:	0.000	4:	2206.495	5:	0.000	6:	87.404
		1:	0.000	2:	0.000	3:	0.000	4:	0.000	5:	0.000	6:	0.000
7:	-1.126	1:	0.000	2:	21.886	3:	0.000	4:	32.748	5:	0.000	6:	1.890
		1:	0.000	2:	0.000	3:	0.000	4:	0.000	5:	0.000	6:	0.000
8:	275.112	1:	0.000	2:	-193.100	3:	0.000	4:	15.878	5:	0.000		
		1:	0.000	2:	0.000	3:	0.000	4:	0.000	5:	0.000		
9:	-2.888	1:	0.000	2:	-135.326	3:	0.000	4:	-30.307				
		1:	0.000	2:	0.000	3:	0.000	4:	0.000				
10:	-23.133	1:	0.000	2:	19.582	3:	0.000						
		1:	0.000	2:	0.000	3:	0.000						
11:	-13.261	1:	0.000	2:	173.354								
		1:	0.000	2:	0.000								
12:	-3.673	1:	0.000										
		1:	0.000										
13:	22.629												

$$\left(\frac{r_0'}{r_0}\right)^n = \left(\frac{0.015}{0.050}\right)^n = \left(\frac{3}{10}\right)^n$$

LAPACK *dgelss*

www.netlib.org/lapack

```
COEFFICIENTS DU DHS DE Bx@r0=0.015 (PREMIER EN T, LES AUTRES EN ppm) :

  0:   1.000000

  1:      0.000   1:      0.000
                  1:      0.000

  2:      0.000   1:      0.000   2:      0.000
                  1:      0.000   2:      0.000

  3:      0.000   1:      0.000   2:      0.000   3:      0.000
                  1:      0.000   2:      0.000   3:      0.000

  4:   -194.715   1:      0.000   2:     97.357   3:      0.000   4:     57.596
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000

  5:      0.000   1:      0.000   2:      0.000   3:      0.000   4:      0.000   5:      0.000
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000   5:      0.000

  6:      0.656   1:      0.000   2:     -0.410   3:      0.000   4:      1.602   5:      0.000   6:      0.041
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000   5:      0.000   6:      0.000

  7:      0.000   1:      0.000   2:      0.003   3:      0.000   4:      0.000   5:      0.000   6:      0.000
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000   5:      0.000   6:      0.000

  8:      0.018   1:      0.000   2:     -0.012   3:      0.000   4:      0.003   5:      0.000
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000   5:      0.000

  9:      0.000   1:      0.000   2:     -0.009   3:      0.000   4:      0.000
                  1:      0.000   2:      0.000   3:      0.000   4:      0.000

 10:      0.000   1:      0.000   2:      0.000   3:      0.000
                  1:      0.000   2:      0.000   3:      0.000

 11:     -0.001   1:      0.000   2:      0.010
                  1:      0.000   2:      0.000

 12:      0.000   1:      0.000
                  1:      0.000

 13:      0.001
```